Lecture 1. Positivity principle

(*) $-\Delta u + V(x)u = f(x)$ in Ω — Schrödinger equation

$\Omega \subseteq \mathbb{R}^N$ $(N \geq 2)$ — domain = open, connected set

Examples: $\Omega = \mathbb{R}^N$ or $\mathbb{R}^2$, $\mathbb{R}^N \setminus \{0\}$, $\mathbb{R}^N \setminus \overline{B_R}$

$B_R(x)$ — open ball, $B_R = B_R(0)$

$-\Delta u - \frac{Z}{|x|} u = f$ — Hydrogen atom model

Assume $V \in L^{\frac{N}{2}+\varepsilon}_{loc}(\Omega)$

$f \in L^p_{loc}(\Omega) \Leftrightarrow \int_{B_R(x)} |f|^p < \infty \quad \forall\, \overline{B_R}(x) \subset \Omega$

$$f \in L^1_{loc}(\Omega)$$

$$V = V^+ - V^-, \quad V^+ = \max\{V, 0\} \text{ - positive part}$$

$$V^- = -\min\{V, 0\} \geq 0 \text{ - negative part}$$

$$H^1_{loc}(\Omega) = \left\{ u \in L^1_{loc}(\Omega) \;\middle|\; \int_{B_R(x)} |\nabla u|^2 + \int_{B_R(x)} |u|^2 < \infty, \; \overline{B_R(x)} \subset \Omega \right\}$$

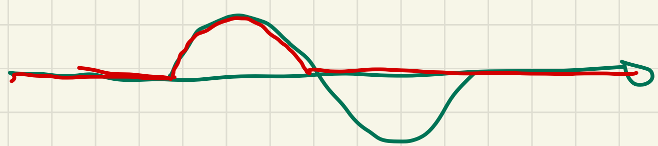
Weak solution of (*) is

$$u \in H^1_{loc}(\Omega) \cap L^1_{loc}(\Omega, V dx) \text{ and}$$

$$\int_\Omega \nabla u \nabla \varphi + \int_\Omega V(x) u \varphi = \int_\Omega f \varphi$$

$$\forall \varphi \in H^1_c(\Omega) \cap L^\infty_c(\Omega)$$

Reasoning: $\varphi \in C^\infty(\Omega) \not\Rightarrow \varphi^+ \in C^\infty(\Omega)$



Weak sub/supersolution if:

$$\underbrace{\int \nabla u \nabla \varphi}_{=\int -\Delta u \varphi} + \int V u \varphi \quad \underbrace{\le}_{\text{sub}} / \underbrace{\ge}_{\text{super}} \int f \varphi \quad \forall\, 0 \le \varphi \in H^1_c \cap L^\infty_c(\Omega)$$

Green identity: $$\int_\Omega \nabla u \nabla \varphi = \int_\Omega (-\Delta u)\varphi + \int_{\partial\Omega} \frac{\partial u}{\partial \nu} \varphi ,$$

$$u, \varphi \in C^2(\Omega)$$

Variational point of view:

$$E_V(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 + \int_\Omega V(x)u^2 - \int_\Omega f(x)u$$

~ the energy that corresponds to $(*)$

Directional derivative of E_V at u in direction φ is

$$\frac{d}{dt} E(u+t\varphi)\Big|_{t=0} = \lim_{t\to 0} \frac{E(u+t\varphi) - E(u)}{t} =$$

Ex. 1

$$= \int_\Omega \nabla u \nabla \varphi + \int_\Omega V u \varphi - \int_\Omega f\varphi = 0 \text{ if}$$

$$\forall \varphi \in C_0^\infty(\Omega)$$

u is a minimum

Regularity (Gilbarg-Trudinger)

1. Local regularity

$$-\Delta u + V(x)u = f \quad , \quad f, V \in L^{\frac{N}{2}+\varepsilon}_{loc}(\Omega)$$

$\Rightarrow u \in C^{1,\alpha}_{loc}(\Omega)$ – ∇u is locally Hölder

2. weak Harnack inequality

$$-\Delta u + V(x) u \geq 0 \text{ in } \Omega \quad , \quad V \in L^{\frac{N}{2}+\varepsilon}_{loc}(\Omega)$$

$\oplus \quad u \geq 0$.

$$\Rightarrow \inf_{B_R(x)} u \geq \frac{c}{R^N} \int_{B_R(x)} u \quad , \quad \forall \overline{B_R(x)} \subset \Omega$$

$$\Rightarrow \frac{1}{u} \in L^\infty_{lo}$$

AAP (Agmon-Allegretto-Piepenbrink positivity principle)

Theorem (AAP-principle) Assume $f \geq 0$.

If $\exists\, u_* > 0$ in Ω such that

$-\Delta u_* + V(x) u_* \geq f(x)$ in Ω then

$$E_V(\varphi) \geq \underbrace{\int_\Omega \left|\nabla \frac{\varphi}{u_*}\right|^2 u_*^2}_{\geq 0} + \underbrace{\int \frac{f}{u_*} \varphi^2}_{\geq 0} \quad \forall \varphi \in C_0^\infty(\Omega)$$

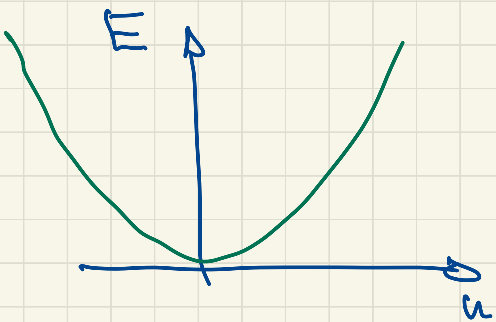
Corollary (nonexistence principle) $f = 0$

Assume $\exists\, \varphi_0 \in C_0^\infty(\Omega)$ such that $E_V(\varphi) < 0$.

$\Longrightarrow$ there is no positive supersolutions to $(*)$.

Hydrogen example: $\underbrace{\int |\nabla u|^2 - \int \frac{Z}{|x|} u^2}_{E_V(u)}$

If $E_V(u) \geq 0$



We may hope to find a minimizer for E_V $\Rightarrow$ a solution to $(*)$.

Assume $u_* > 0$ is a supersol. of $(*)$ and $f \geq 0$.

Take $\psi = \frac{\varphi^2}{u_*}$ as a test function, $\forall \varphi \in C_c^\infty(\Omega)$

$$\nabla \psi = \nabla(\varphi^2 u_*^{-1}) = 2\varphi \nabla \varphi \, u_*^{-1} + \varphi^2 (-u_*^{-2} \nabla u_*) =$$

$$= 2 \underbrace{\frac{\varphi}{u_*}}_{\in L^\infty} \nabla \varphi - \underbrace{\frac{\varphi^2}{u_*^2}}_{\in L^\infty} \underbrace{\nabla u_*}_{\in H^1_{loc}}$$

— we can use ψ as a test function

$$\int \nabla u_* \nabla \psi + \int V u_* \psi =$$

$$= \int 2 \nabla u_* \nabla \varphi \, \frac{\varphi}{u_*} - \int |\nabla u_*|^2 \frac{\varphi^2}{u_*^2} + \int V \cancel{u_*} \frac{\varphi^2}{\cancel{u_*}}$$

(since u_* is a supersol.) $\geq \int \delta \, \cancel{u_*} \frac{\varphi^2}{\cancel{u_*}}$ $\quad (**)$

$$\int |\nabla \varphi|^2 + V\varphi^2 - \int \left|\nabla \frac{\varphi}{u_*}\right|^2 u_*^2 =$$

$$= \int |\nabla \varphi|^2 + V\varphi^2 - \int \left(\frac{|\nabla\varphi|^2}{u_*^2} - 2\varphi\nabla\varphi \frac{\nabla u_*}{u_*^3} + \varphi^2 \frac{|\nabla u_*|^2}{u_*^4} \right) u_*^2 =$$

$$= \int \cancel{|\nabla\varphi|^2} + V\varphi^2 - \int \cancel{|\nabla\varphi|^2} - 2\varphi\nabla\varphi \frac{\nabla u_*}{u_*} + \varphi^2 \frac{|\nabla u_*|^2}{u_*^2} \geq$$

by (**) $\geq \int f \varphi^2$.

$$\Rightarrow \int |\nabla\varphi|^2 + V\varphi^2 \geq \underbrace{\int \left|\nabla \frac{\varphi}{u_*}\right|^2 u_*^2}_{\geq 0} + \int f\varphi^2$$

$\blacksquare$

Elementary proof of Hardy inequality:

$$\int_{\mathbb{R}^N} |\nabla\varphi|^2 \geq C_H \int_{\mathbb{R}^N} \frac{\varphi^2}{|x|^2} \quad \forall \varphi \in C_0^\infty(\mathbb{R}^N)$$

$N \geq 3$

$C_H = \left(\frac{N-2}{2}\right)^2$ – is a sharp constant.

$$E_V(\varphi) = \frac{1}{2}\int |\nabla\varphi|^2 - C_H \int \frac{\varphi^2}{|x|^2} \geq 0 \quad \forall \varphi \in C_c^\infty(\mathbb{R}^N)$$

The corresponding Schrödinger eqn.

$$-\Delta u - \frac{C_H}{|x|^2} u \geq 0,$$

We only need to find $u_* > 0$!

Take $u_* = |x|^{-\frac{N-2}{2}}$. Recall "radial" representation of $-\Delta$:

$$-\Delta u(r) = -\frac{\partial^2}{\partial r^2} u(r) - \frac{N-1}{r}\frac{\partial}{\partial r} u(r)$$

If $u_*(r) = r^{-\frac{N-2}{2}}$ $\quad r = |x|$ then

$$-\Delta u_* = \left(\frac{N-2}{2}\right)^2 \underbrace{r^{-\frac{N-2}{2}}}_{u_*}{}^{-2} = \left(\frac{N-2}{2}\right)^2 r^{-2} u_*$$

$$\Rightarrow \quad -\Delta u_* - \left(\frac{N-2}{2}\right)^2 u_* = 0 \quad \text{in } \mathbb{R}^N \setminus \{0\}$$

but $u_* = r^{-\frac{N-2}{2}} \notin H^1_{loc}(B_1)$

$\Rightarrow$ Compute ∇u_* and see that $\int_{B_1} |\nabla u_*|^2 = +\infty$

$$\frac{1}{\omega_N}\int_0^1 \underbrace{\left|\frac{\partial}{\partial r}\left(r^{-\frac{N-2}{2}}\right)\right|^2 r^{N-1}\,dr}_{=+\infty} = \int_{B_1} |\nabla u_*|^2$$

Exercise:
compute this!

What we actually proved is that

$$\int |\nabla u|^2 \geq C_H \int \frac{u^2}{|x|^2} \quad \forall\, u \in C_0^\infty(\mathbb{R}^N \setminus \{0\})$$

Is the inequality valid for all $u \in C_0^\infty(\mathbb{R}^N)$?